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Nonlinear mode decoupling for classes of evolution equations

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Abstract. The method of third-order mode coupling is applied to a general evolution equation, which includes as special cases many of the modified and higher-order Korteweg–de Vries equations which have recently appeared in the literature. The equations for the slow changes in the amplitudes and phases are derived and specialised to two different classes of evolution equations. The first class exhibits the property of non-resonant mode decoupling, the evolution of each wave being governed only by its own parameters. Included in this class are the KdV equation, some of the mKdV equations and the Sharma–Tasso–Olver equation. Equations of the second class leave the amplitudes and hence the energetics of the waves constant as long as the mode coupling is non-resonant. Here one finds the fifth-order KdV equations, besides the KdV and STO equations.

1. Introduction

In a previous paper (Verheest and Eeckhout 1977) the Korteweg–de Vries equation was found to possess the remarkable property that in a perturbation scheme, in which the linear term is a superposition of plane waves, the waves also decouple in third order. The slow time variation of each wave is then determined only by the parameters (amplitude and wavenumber) of that wave itself, as if it were the only wave present at this stage in the perturbation scheme. Such a kind of nonlinear superposition is rather unusual, hence the search for other PDEs with a similar property. In a subsequent paper (Verheest and Hereman 1979) some other equations were found, besides the KdV equation, starting from a general class of PDEs in which each term was linear in the space derivatives, although the coefficients were functions of the dependent variable. Recently a great number of new nonlinear evolution equations have appeared in the literature (some examples will be given in the next section), in the wake of the renewed interest in the KdV equation and generalisations since the pioneering work of Gardner *et al* (1967). Most of these evolution equations or higher-order modified KdV equations are, however, nonlinear in the derivatives and it seemed thus worthwhile to rework the analysis, starting from a more general PDE which would include most of the equations now being studied by other means or for other purposes.

A perturbation scheme is used, in which the dependent variable is expanded together with a two-timescale approach. In the resulting set of equations the first one is linear and for the solution a superposition of plane waves is taken, which amounts to some kind of Fourier analysis, and is hence rather general. The requirement that the expansion be non-secular then leads in third order to equations giving the slow variations in amplitude and phase of the waves. In general these are all coupled, except

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for certain classes of equations which are found from the decoupling condition. If the amplitude equations would not decouple, the number of waves in the linear superposition becomes of paramount importance. This could introduce, on purely theoretical grounds, an element of ambiguity in the method of wave interaction in third order, as the number of waves is not always a simple quantity to give, especially where some of the waves in a nonlinear interaction may grow out of the noise, with initially a negligible amplitude. Such a difficulty can only be avoided if one is sure to launch in a given experimental situation precisely three or four waves with frequencies and wavenumbers satisfying the appropriate third-order selection rules. Finally, classes of nonlinear evolution equations are given for which the amplitudes remain constant in third order, which means that such nonresonant interaction cannot change the wave energy.

2. General formalism and third-order amplitude and phase equations

As classes of evolution equations we consider the following nonlinear PDE in one dependent variable u:

$$u_t + pu_{xxx} + qu_{xxxxx} = (A_1 u + A_2 u^2)u + (B_1 u + B_2 u^2)u_x + (C_1 u + C_2 u^2)u_{xx} + (D_1 u + D_2 u^2)u_{xxx} + (E_0 + E_1 u)u_x^2 + (F_0 + F_1 u)u_x u_{xx} + G_0 u_x^3 \tag{1}$$

where p, q, A_1 to G_0 are constants.

The derivatives with respect to space or time have been indicated with the corresponding subscripts x or t. Equation (1) is the most general one which includes up to three space derivatives per term, except for the linear part where a fifth derivative was included. Nonlinearities of higher order than cubic have been omitted, as later on we shall adopt a perturbation scheme in which we investigate effects up to third order only.

The linear part of (1) has been reduced to its most simple form through a judicious rescaling in x and t. As a consequence, a term in u_x, if occurring, can be eliminated, and for p and q only the following possibilities remain:

$$p = -1, 0, +1 \qquad q = 1 \qquad \text{or} \qquad p = 1 \qquad q = 0. \tag{2}$$

The motivation for studying a general equation of the type (1) lies in the fact that it includes a great many examples of nonlinear evolution equations, which have appeared recently in the literature. Included as special cases in (1) are the KdV equation itself, the modified KdV equations

$$u_t + \alpha u u_x + \beta u^2 u_x + u_{xxx} = 0 \tag{3}$$

(Driscoll and O'Neil 1976), the potential KdV or mKdV equations

$$u_t + u_{xxx} + \alpha u_x^3 + \beta u_x^2 = 0 \tag{4}$$

(Fokas 1980), the Sharma–Tasso–Olver equation

$$u_t + 3u_x^2 + 3u^2 u_x + 3u u_{xx} + u_{xxx} = 0 \tag{5}$$

(Sharma and Tasso 1977, Olver 1977), the KdV equation with higher-order dispersion

$$u_t + u u_x + u_{xxx} \pm u_{xxxxx} = 0 \tag{6}$$

(Kodama and Taniuti 1978), and then a spate of fifth-order KdV equations, which cannot be rescaled into each other, such as the Sawada–Kotera or Caudrey–Dodd–

Gibbon equation

$$u_t + 5u^2 u_x + 5u_x u_{xx} + 5uu_{xxx} + u_{xxxxx} = 0 \tag{7}$$

(Sawada and Kotera 1974, Caudrey *et al* 1976), the Lax equation

$$u_t + 30u^2 u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{xxxxx} = 0 \tag{8}$$

(Lax 1968), the Kaup–Kupershmidt equation

$$u_t + 20u^2 u_x + 25u_x u_{xx} + 10uu_{xxx} + u_{xxxxx} = 0 \tag{9}$$

(Hirota and Ramani 1980, Fordy and Gibbons 1980) and the higher-order Sawada–Kotera equation

$$u_t + 2u^2 u_x + 6u_x u_{xx} + 3uu_{xxx} + u_{xxxxx} = 0 \tag{10}$$

(Ito 1980).

We now expand the dependent variable u in terms of some small parameter ε,

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \ldots \tag{11}$$

and use a two-time-scale approach

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t_0} + \varepsilon^2 \frac{\partial}{\partial t_2} \qquad t_0 = t \qquad t_2 = \varepsilon^2 t. \tag{12}$$

This is similar to what was done before, for other types of equations (Verheest and Eeckhout 1977, Verheest and Hereman 1979). The ordering used precludes phenomena on the $t_1 = \varepsilon t$ time scale. Such phenomena are of the three-wave interaction type, have been extensively studied in the literature and are always resonant, hence irrelevant in the context of the present study. The precise determination of ε will have to be made on physical grounds in a given situation. As a single example, the study of third-harmonic generation in nonlinear optics (Armstrong *et al* 1962), the ratio of the cubic to the linear susceptibilities determines the expansion scheme. Without a quadratic susceptibility only third-order interactions are possible. Slow spatial scales have not been included, because this conceptually leads to similar conclusions as drawn later on, at the price, however, of much more involved calculations.

There is no zeroth-order term in (11), because it is always possibile to eliminate such a term by a suitable shift in u, together with a redefinition of the coefficients occurring in (1). The set of equations which replaces (1) is then found to be

$$Lu_1 = 0$$

$$Lu_2 = A_1 u_1^2 + B_1 u_1 \frac{\partial u_1}{\partial x} + C_1 u_1 \frac{\partial^2 u_1}{\partial x^2} + D_1 u_1 \frac{\partial^3 u_1}{\partial x^3} + E_0 \left(\frac{\partial u_1}{\partial x}\right)^2 + F_0 \frac{\partial u_1}{\partial x} \frac{\partial^2 u_1}{\partial x^2}$$

$$\begin{aligned} Lu_3 = {} & -\frac{\partial u_1}{\partial t_2} + 2A_1 u_1 u_2 + A_2 u_1^3 + B_1 u_1 \frac{\partial u_2}{\partial x} + B_1 u_2 \frac{\partial u_1}{\partial x} + B_2 u_1^2 \frac{\partial u_1}{\partial x} + C_1 u_1 \frac{\partial^2 u_2}{\partial x^2} \\ & + C_1 u_2 \frac{\partial^2 u_1}{\partial x^2} + C_2 u_1^2 \frac{\partial^2 u_1}{\partial x^2} + D_1 u_1 \frac{\partial^3 u_2}{\partial x^3} + D_1 u_2 \frac{\partial^3 u_1}{\partial x^3} + D_2 u_1^2 \frac{\partial^3 u_1}{\partial x^3} \\ & + 2E_0 \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial x} + E_1 u_1 \left(\frac{\partial u_1}{\partial x}\right)^2 + F_0 \frac{\partial u_1}{\partial x} \frac{\partial^2 u_2}{\partial x^2} \\ & + F_0 \frac{\partial u_2}{\partial x} \frac{\partial^2 u_1}{\partial x^2} + F_1 u_1 \frac{\partial u_1}{\partial x} \frac{\partial^2 u_1}{\partial x^2} + G_0 \left(\frac{\partial u_1}{\partial x}\right)^3. \end{aligned} \tag{13}$$

Here the linear operator L is defined as

$$L = \frac{\partial}{\partial t_0} + p\frac{\partial^3}{\partial x^3} + q\frac{\partial^5}{\partial x^5}. \tag{14}$$

The first-order solution u_1 is taken as a superposition of N plane waves

$$\begin{aligned} u_1 &= \sum_{j=1}^{N} a_j(t_2)\cos\left[k_j x + (pk_j^3 - qk_j^5)t_0 + \alpha_j(t_2)\right] \\ &\equiv \sum_{j=1}^{N} a_j(t_2)\cos\phi_j(x, t_0, t_2) \end{aligned} \tag{15}$$

where in the phase of each wave the particular dispersion corresponding to L has already been taken into account. As the first equation (13) is now satisfied, the second equation (13) yields

$$\begin{aligned} Lu_2 = \tfrac{1}{2}\sum_{l,m=1}^{N}\sum \{ &(B_1 k_m - D_1 k_m^3 - F_0 k_l^2 k_m)[\sin(\phi_l - \phi_m) - \sin(\phi_l + \phi_m)] \\ &+ (A_1 - C_1 k_m^2 - E_0 k_l k_m)\cos(\phi_l + \phi_m) \\ &+ (A_1 - C_1 k_m^2 + E_0 k_l k_m)\cos(\phi_l - \phi_m)\} a_l a_m. \end{aligned} \tag{16}$$

The avoidance of secular terms in u_2 requires that

$$A_1 = 0 \qquad C_1 = E_0. \tag{17}$$

The solution of (16) is then

$$\begin{aligned} u_2 = \tfrac{1}{2}\sum_{l=1}^{N} (\mu_{ll}^{+}\sin 2\phi_l + \nu_{ll}^{+}\cos 2\phi_l)a_l^2 + \sum_{l<m=1}^{N}\sum &[\mu_{lm}^{+}\sin(\phi_l + \phi_m) + \mu_{lm}^{-}\sin(\phi_l - \phi_m) \\ &+ \nu_{lm}^{+}\cos(\phi_l + \phi_m) + \nu_{lm}^{-}\cos(\phi_l - \phi_m)]a_l a_m \end{aligned} \tag{18}$$

if

$$\begin{aligned} \mu_{lm}^{\pm} &= \pm\frac{C_1(k_l \pm k_m)}{2k_l k_m[3p - 5q(k_l^2 \pm k_l k_m + k_m^2)]} \\ \nu_{lm}^{\pm} &= \frac{\mp B_1 \pm D_1(k_l^2 \mp k_l k_m + k_m^2) + F_0 k_l k_m}{2k_l k_m[3p - 5q(k_l^2 \pm k_l k_m + k_m^2)]}. \end{aligned} \tag{19}$$

Substitution of the form (15) for u_1 and (18) for u_2 into the last equation (13) gives

$$\begin{aligned} Lu_3 = -\sum_{j=1}^{N}\left(\frac{\partial a_j}{\partial t_2}\cos\phi_j - a_j\frac{\partial \alpha_j}{\partial t_2}\sin\phi_j\right) + \tfrac{1}{4}\sum_{j,l,m=1}^{N}\sum\sum &a_j a_l a_m[\hat{\gamma}_{jlm}\cos(\phi_j + \phi_l + \phi_m) \\ &+ \gamma_{jlm}^{+}\cos(-\phi_j + \phi_l + \phi_m) + 2\gamma_{jlm}^{-}\cos(\phi_j - \phi_l + \phi_m) \\ &+ \delta_{jlm}^{+}\sin(\phi_j + \phi_l + \phi_m) + \xi_{jlm}^{+}\sin(-\phi_j + \phi_l + \phi_m) \\ &+ \xi_{jlm}^{-}\sin(\phi_j - \phi_l + \phi_m) + \delta_{jlm}^{-}\sin(\phi_j + \phi_l - \phi_m)] \\ &+ \tfrac{1}{2}\sum_{j,l=1}^{N}\sum a_j a_l^2 k_j \nu_{ll}^{-}[k_j C_1\cos\phi_j + (B_1 - D_1 k_j^2)\sin\phi_j]. \end{aligned} \tag{20}$$

Use of the following abbreviations has been made:

$$\begin{aligned}
\hat{\gamma}_{jlm} &= (k_j+k_l+k_m)[\mu^+_{lm}\eta^+_{jlm} - C_1\nu^+_{lm}(k_j+k_l+k_m)] + A_2 - C_2k_j^2 - E_1k_lk_m \\
\gamma^\pm_{jlm} &= (\mp k_j \pm k_l + k_m)[\pm\mu^\pm_{lm}\zeta^\pm_{jlm} - C_1\nu^\pm_{lm}(\mp k_j \pm k_l + k_m)] + A_2 - C_2k_j^2 \mp E_1k_lk_m \\
\delta^\pm_{jlm} &= -(k_j+k_l\pm k_m)[\nu^\pm_{lm}\eta^\pm_{jlm} + C_1\mu^\pm_{lm}(k_j+k_l\pm k_m)] \\
&\qquad \mp B_2k_m \pm D_2k_m^3 \pm F_1k_l^2k_m \pm G_0k_jk_lk_m \\
\xi^\pm_{jlm} &= (\mp k_j \pm k_l + k_m)[-\nu^\pm_{lm}\zeta^\pm_{jlm} \mp C_1\mu^\pm_{lm}(\mp k_j \pm k_l + k_m)] \\
&\qquad - B_2k_m + D_2k_m^3 + F_1k_l^2k_m - G_0k_jk_lk_m
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
\eta^\pm_{jlm} &= B_1 - D_1[k_j^2 - k_j(k_l\pm k_m) + (k_l\pm k_m)^2] - F_0k_j(k_l\pm k_m) \\
\zeta^\pm_{jlm} &= B_1 - D_1[k_j^2 + k_j(k_l\pm k_m) + (k_l\pm k_m)^2] + F_0k_j(k_l\pm k_m).
\end{aligned} \tag{22}$$

As $\cos\phi_j$ and $\sin\phi_j$ belong to the kernel of L, these terms will contribute to secularities in u_3, if left in (20). The requirement that all such terms disappear from (20) gives the amplitude and phase equations in third order:

$$\begin{aligned}
\frac{\partial a_j}{\partial t_2} &= \tfrac{1}{2}a_j \sum_{l\neq j} a_l^2[k_j(\mu^+_{lj}\zeta^+_{llj} - \mu^-_{lj}\zeta^-_{llj}) - C_1k_j^2(\nu^+_{lj}+\nu^-_{lj}) + 3A_2 - 2C_2k_l^2 - C_2k_j^2 + E_1k_l^2] \\
&\qquad + \tfrac{1}{4}a_j^3(k_j\mu^+_{jj}\zeta^+_{jjj} - C_1k_j^2\nu^+_{jj} + 3A_2 - 3C_2k_j^2 + E_1k_j^2) \\
\frac{\partial \alpha_j}{\partial t_2} &= \tfrac{1}{2}k_j \sum_{l\neq j} a_l^2[\nu^+_{lj}\zeta^+_{llj} + \nu^-_{lj}\zeta^-_{llj} + C_1k_j(\mu^+_{lj} - \mu^-_{lj}) + B_2 - D_2k_j^2 - F_1k_l^2 + 3G_0k_l^2] \\
&\qquad + \tfrac{1}{4}a_j^2k_j(\nu^+_{jj}\zeta^+_{jjj} + C_1k_j\mu^+_{jj} + B_2 - D_2k_j^2 - F_1k_j^2 + 3G_0k_j^2).
\end{aligned} \tag{23}$$

3. Nonlinear decoupling

The N waves in u_1 will be said to decouple in third order whenever it is possible to reduce (23) to

$$\begin{aligned}
\frac{\partial a_j}{\partial t_2} &= \tfrac{1}{4}a_j^3[k_j\mu^+_{jj}\zeta^+_{jjj} - C_1k_j^2\nu^+_{jj} + 3A_2 + (E_1 - 3C_2)k_j^2] \\
\frac{\partial \alpha_j}{\partial t_2} &= \tfrac{1}{4}k_ja_j^2[\nu^+_{jj}\zeta^+_{jjj} + C_1k_j\mu^+_{jj} + B_2 + (3G_0 - D_2 - F_1)k_j^2]
\end{aligned} \tag{24}$$

because then the slow variations in amplitude and phase of each wave are only determined by the parameters (amplitude and wavenumber) of that wave itself, regardless of whatever other waves are present. Such a decoupling requirement gives the following set of restrictions on the possible coefficients in (1):

$$\begin{aligned}
&A_1 = 0 \qquad A_2 = -\tfrac{1}{9}B_1C_1 \qquad q(5B_1 - 3pD_1) = 0 \\
&B_2 = -\tfrac{1}{5}qD_1^2 + \frac{q-1}{3}(C_1^2 + 2B_1D_1) \qquad qC_1 = 0 \qquad E_0 = C_1 \qquad C_2 = -C_1D_1 \\
&D_2 = \tfrac{2}{3}(q-1)D_1^2 \qquad E_1 = -\tfrac{1}{3}C_1(F_0 + 3D_1) \qquad q(F_0 - 2D_1) = 0 \\
&F_1 = 3G_0 + \tfrac{1}{3}(1-q)F_0(F_0 - 3D_1).
\end{aligned} \tag{25}$$

This leads essentially to two different classes of nonlinear evolution equations, according to whether q vanishes or not. As a first class of equations we find

$$u_t+u_{xxx}=-\tfrac{1}{9}\beta\gamma u^3+[\beta u-\tfrac{1}{3}(\gamma^2+2\beta\delta)u^2]u_x+\gamma u(1-\delta u)u_{xx}+\delta u(1-\tfrac{2}{3}\delta u)u_{xxx}$$
$$+\gamma[1-(\delta+\tfrac{1}{3}\xi)u]u_x^2+\{\xi+[3\zeta+\tfrac{1}{3}\xi(\xi-3\delta)]u\}u_xu_{xx}+\zeta u_x^3 \qquad (26)$$

where β, γ, δ, ξ and ζ are constant parameters. The corresponding amplitude and phase equations are

$$\frac{\partial a_j}{\partial t_2}=\frac{\gamma}{24}a_j^3[\beta+(\xi-3\delta)k_j^2]$$
$$\frac{\partial \alpha_j}{\partial t_2}=-\frac{1}{24k_j}a_j^2(\beta-\delta k_j^2)[\beta+(\xi-3\delta)k_j^2]. \qquad (27)$$

Included in (26) are the KdV equation, the Sharma–Tasso–Olver equation (5) (for the choice $\gamma=-3$, $\beta=\delta=\xi=\zeta=0$) and the modified KdV equations of the type

$$u_t+u_{xxx}=(3u+6u^2)(\kappa u_x-u_{xxx}) \qquad \kappa=-1,0,+1 \qquad (28)$$

discussed earlier (Verheest and Hereman 1979). The other mKdV equations appearing in (3) or (4) do not possess this property of third-order decoupling.

From (27) the square of the amplitude is found to be

$$a_j^2(t_2)=\frac{a_j^2(0)}{1-a_j^2(0)\frac{1}{12}\gamma[\beta+(\xi-3\delta)k_j^2]t_2}. \qquad (29)$$

This allows the computation of the phase α_j if need be. The behaviour of $a_j^2(t_2)$ will depend upon the sign of $\gamma[\beta+(\xi-3\delta)k_j^2]$. If this expression is positive, an explosive instability occurs in a time τ_j (on the slow time scale t_2) given by

$$\tau_j=\{a_j^2(0)\tfrac{1}{12}\gamma[\beta+(\xi-3\delta)k_j^2]\}^{-1} \qquad (30)$$

but when $\gamma[\beta+(\xi-3\delta)k_j^2]$ is negative, the wave amplitude decays always to zero in an infinite time. In both cases either the waves or the medium in which the waves propagate dissipate all their energy, as is characteristic for non-conservative systems. A last remark about this class of equations concerns the possible invariance of (26) for space–time reversal. Equations which are invariant for such a reversal automatically lead to constant amplitudes in non-resonant third-order interaction (Verheest 1980). (26) is invariant as soon as γ vanishes. But (27) also shows that the wave amplitudes (and the phases, incidentally) remain constant when γ is not zero, if β vanishes and $\xi=3\delta$. This corresponds with equations which are not invariant for space–time reversal and yet leave the wave amplitudes constant. The Sharma–Tasso–Olver equation (5) is one of those. The condition that an equation is invariant for space–time reversal hence is sufficient but not necessary to obtain constant amplitudes.

Returning now to (25), we find a second class of equations with $q\neq 0$

$$u_t+pu_{xxx}+u_{xxxxx}=\tfrac{1}{5}\delta(3p-\delta u)uu_x+\delta uu_{xxx}+(2\delta+3\zeta u)u_xu_{xx}+\zeta u_x^3$$
$$p=-1,0,+1 \qquad (31)$$

where δ and ζ are constant parameters. The corresponding amplitude and phase equations are

$$\frac{\partial a_j}{\partial t_2}=0 \qquad \frac{\partial \alpha_j}{\partial t_2}=-\frac{\delta^2}{200k_j}(3p-5k_j^2)a_j^2. \qquad (32)$$

None of the fifth-order KdV equations (6)–(10) shows the property of third-order decoupling. As the equations of the type (31) are all invariant for a space–time reversal, the wave amplitudes indeed remain constant and hence the phases increase or decrease linearly in time.

4. Constant wave amplitudes

In the previous section we have found an example of a nonlinear equation which in third order gives waves that either are explosively unstable or decay to zero. Systems described by such equations act as energy sources or sinks, and this may give rise to difficulties in interpretation when the total number of waves is not specified, as would seem the case here. Hence one would expect it to be more natural or plausible to find systems of a conservative nature, in the sense that the non-resonant third-order interactions between some waves do not change the energy or the amplitude of each wave. Hence it is worthwhile to investigate in this section which classes of equations lead to constant amplitudes in third order. One finds three different types of such equations:

$$\text{(i)} \qquad u_t + u_{xxx} = \beta u^2 u_x + \gamma u(1-\delta u)u_{xx} + (\delta + \delta' u)uu_{xxx} + \gamma(1-2\delta u)u_x^2 + (3\delta + \xi u)u_x u_{xx} + \zeta u_x^3 \qquad (33)$$

$$\text{(ii)} \qquad u_t + pu_{xxx} + qu_{xxxxx} = (\beta + \beta' u)uu_x + (\delta + \delta' u)uu_{xxx} + (\xi + \xi' u)u_x u_{xx} + \zeta u_x^3$$

$$\text{(iii)} \qquad u_t + pu_{xxx} + u_{xxxxx} = \beta u^2 u_x + \gamma uu_{xx} + \delta u^2 u_{xxx} + \gamma u_x^2 + \xi uu_x u_{xx} + \zeta u_x^3$$

where β, β', γ, δ, δ', ξ, ξ' and ζ are constant parameters. The middle equation in (33) is invariant for space–time reversal, the two others are not if $\gamma \neq 0$. The Sharma–Tasso–Olver equation (5) belongs to the type (i) with $\beta = \gamma = -3$ and $\delta = \delta' = \xi = \zeta = 0$. The invariant equation of the type (ii) includes as special cases the KdV equation itself, the mKdV equation (3), the KdV equation with higher-order dispersion (6) and the fifth-order KdV equations (7) to (10).

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